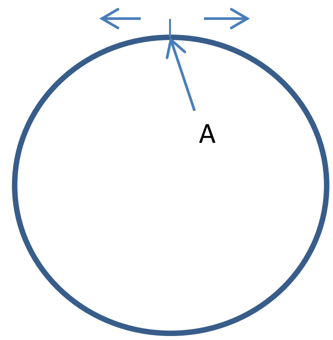
Introduction

While using smartphones you may have noticed an icon indicating ‘system busy’. This is a small circle in which a rotation of a point is animated. If closely watched you may notice that actually there are two points rotating through the border of the circle, one in clock-wise direction and other in anti-clock-wise direction. Naturally while rotating these two points meet several times. While experimenting I reached at some interesting mathematical features of this. Read on.

Study of Unit Length Circle



Consider a circle having unit length. A point ‘A’ is considered on the top of the circle. The points on the circle will have a one-to-one correspondence with semi-open interval [0, 1). Point A is marked with 0. Two objects say x and y are placed at A. Object x is moved in clockwise direction and y is moved in counter clockwise direction. Their speeds are X and Y respectively. These objects will meet at various points on the circle. The study of this seems to be interesting.

Suppose the speeds of x and y are 1 and 2 respectively. After point A their first meeting point will be 1/3 units in clock-wise direction from A. The second meeting point will be 2/3 units in clock-wise direction from A. The third meeting point will be at A itself. So objects x and y repeatedly meets only at three points. For the ongoing discussion we consider speed of x is fixed at 1 unit per second and speed of y is some other quantity.

Study of meeting points with speed of $y = \sqrt{2}$

When Y is $\sqrt{2}$ (or any irrational number like $\sqrt{3}$) the meeting points of x and y will have some special features.

1. **After the starting x and y will never meet again at A. Also every meeting points of x and y will be unique, it will not be repeated.**

Suppose after some interval x and y meet again at A. Suppose that x completed m full rotations and y completed n full rotations. Since the time taken by x and y are same and speeds are 1 and $\sqrt{2}$:
$\frac{m}{1} = \frac{n}{\sqrt{2}}$ (because time = distance/speed) which give $\sqrt{2} = \frac{n}{m}$
This is impossible since m and n are integers. So x and y will never meet again at A. Due to the same argument any meeting points will not be repeated. That is, every time x and y will meet at a unique point.

2. **There will be infinitely many points on the circle that x and y will never meet.**

This is proved with the insight that every meeting point will be an algebraic number between 0 and 1. For example the first meeting point is $\sqrt{2} - 1$. This is an algebraic number. Due to this all their meeting points will be algebraic numbers. But there are infinitely many non- algebraic (transcendental) numbers between 0 and 1. This proves the above statement.

3. **x and y will never meet at any rational point on the circle.**

Suppose x and y meet at a rational point r. Suppose x completed m full rotations and travelled additional length r, and y will complete n rotations if it travels an additional length r. So the distance travelled by x and y are m+r and n-r. Since the time taken by x and y are same and speeds are 1 and $\sqrt{2}$, we get $\frac{m+r}{1} = \frac{n-r}{\sqrt{2}}$. This give $\sqrt{2} = \frac{n-r}{m+r}$ which is impossible since m, n are integers and r is rational. Hence the proof.

So all transcendental numbers as well as rational numbers will be excluded from the meeting points of x and y.

Study of meeting points with different speed of y

4. **Speed of x is fixed at 1 and speed of y we term as Y. Let M be the set of meeting points of x & y when Y = $\sqrt{2}$. Let N be the set of meeting points of x & y when Y = $\sqrt{3}$. Then except 0 (the starting point) there will be no other points common in M and N.**

After 0, the first meeting point when speed of y = $\sqrt{2}$ will be $\sqrt{2} - 1$. The next point will be $2 * (\sqrt{2} - 1)$ and generally it will be $m * (\sqrt{2} - 1) - n$ where m and n are integers and n is an integer which is required to cancel the integer part. For example when m = 100, $m * (\sqrt{2} - 1)$ becomes 41.421356 so n should be 41 so that $m * (\sqrt{2} - 1) - n$ becomes 0.421356.

Similarly when speed of y = $\sqrt{3}$ the first meeting point after 0 will be $\frac{\sqrt{3}-1}{2}$ and further meeting points can be defined with the formula: $p * \frac{\sqrt{3}-1}{2} - q$ where p and q are integers. We can check whether any of these can be a member of M.

So we need to check whether $m * (\sqrt{2} - 1) - n$ can be equal to $p * \frac{\sqrt{3}-1}{2} - q$ for any integer value of m, n, p and q.

$$m * (\sqrt{2} - 1) - n = p\, \frac{\sqrt{3} - 1}{2} - q$$

So $m * (\sqrt{2} - 1) - n = p\frac{\sqrt{3}-1}{2}$ ---- Because when take q from the right side to left side we get –n+q on left side which again is an integer only which we represent with n.

So $m(\sqrt{2} - 1) - n = p\sqrt{3}$ -- Because when we cancel 2 from right side m, n and p are still integers.

Now, $m\sqrt{2} - n = p\sqrt{3}$ or $m\sqrt{2} - p\sqrt{3} = n$ -------- (4.a)
Multiplying both sides with $m\sqrt{2} + p\sqrt{3}$ we get:

$2m^2 - 3p^2 = n(m\sqrt{2} + p\sqrt{3})$ or $\frac{2}{n}m^2 - \frac{3}{n}p^2 = m\sqrt{2} + p\sqrt{3}$

Or $m\sqrt{2} + p\sqrt{3} = k$ where k is a rational number -------- (4.b)

Adding (4.a) and (4.b) we get.

$2m\sqrt{2} \ = k + n$
Where m, n and k are integers or rational numbers. This is impossible and hence the proof.

This finding is naturally extended to: The set when Y = $\sqrt{n}$ and set when Y = $\sqrt{m}$ will not have any common point when $\sqrt{n}$ and $\sqrt{m}$ are irrational and m and n does not have any common factor.

5. **Extension of the set.**

The set of counting numbers start with 1 and count up to 2, 3, 4, …. As a natural logic we can count back from 4, 3, 2, 1 and naturally it extend to 0, -1, -2, -3 … etc. This kind of extension is applicable to the set we have discussed (So far we have not given any name for that. Though the name does not have much significance to the properties, now onwards I am calling it as Unit Length Circle Set or simply ULC set). So ULC set with seed = $\sqrt{2}$ can be extended as below.

The first member is $\sqrt{2} - 1$ = 0.414213562373095
Second member is $2 * (\sqrt{2} - 1)$ = 0.8284271247461901
Third member is $3 * \left(\sqrt{2} - 1\right) - 1$ =0.2426406871192851

So extending backwards from the first member:
0^{th} member = 0
-1^{st} member = $1 - \left(\sqrt{2} - 1\right)$ = 0.585786437626905
-2^{nd} member = $1 - 2 * \left(\sqrt{2} - 1\right)$ = 0.1715728752538099

Note that sum of 1^{st} and -1^{st} members = 1, sum of 2^{nd} and -2^{nd} members = 1, sum of n^{th} and $-n^{th}$ members = 1.

The ULC sets we discussed are having their origin at 0. We can consider ULC sets with non-zero origin also. The following theorem is a natural outcome of that and so simple that I am not giving any proof.

6. **ULC set of seed = $\sqrt{2}$ with origin at 0 and ULC set of $\sqrt{2}$ with origin at r will not have any common member when r is a non-zero rational number.**

This statement means that with $\sqrt{2}$ as seed there are infinite number of disjoint ULC sets corresponding to every rational number between 0 and 1.

Summary: The seed of ULC set can be any irrational number. The sets produced with two separate seeds will be disjoint if these seeds have some properties (I am investigating for a clear statement of this property). In addition to that with each seed we can produce infinite number of disjoint sets by selecting different rational origins in the range [0, 1).

7. **Further investigation of the ULC**

When investigating further we can find that the circle need not be unit length circle. It can be any circle. The speed of the objects also does not have any relevance in connection with the perimeter of the circle. What matters is that **ratio** of the speed of objects. For getting a set of infinite points the ratio can be $1 : \sqrt{2}$ or 1 : $\sqrt{3}$ or similar. When two objects travel in speed ration of $1 : \sqrt{2}$ in opposite directions their meeting points will be infinite and their meeting points never repeat. The perimeter of circle is non-relevant in this. If the perimeter is taken as 1 then we get the first meeting point at $\sqrt{2} - 1$. If we take the perimeter as $\sqrt{2} + 1$ we get the meeting point at $\left(\sqrt{2} - 1\right) * \left(\sqrt{2} + 1\right) = 1$.

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