# Fibonacci sequence and Geometric sequence

Fibonacci series is defined in Wikipedia as below:

The Fibonacci numbers or Fibonacci series or Fibonacci sequence are the numbers in the following integer sequence:
1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, …

or (often, in modern usage):

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, …

By definition, the first two numbers in the Fibonacci sequence are 1 and 1, or 0 and 1, depending on the chosen starting point of the sequence, and each subsequent number is the sum of the previous two.

In mathematical terms, the sequence Fn of Fibonacci numbers is defined by the recurrence relation
$F_n = F_{n-1} + F_{n-2}$
with seed values $F_1 = 1$, $F_2 = 1$ or $F_0 = 0$, $F_1 = 1$.

The Fibonacci sequence is named after Leonardo Fibonacci. His 1202 book Liber Abaci introduced the sequence to Western European mathematics, although the sequence had been described earlier in Indian mathematics. By modern convention, the sequence begins either with F0 = 0 or with F1 = 1. The Liber Abaci began the sequence with F1 = 1, without an initial 0.

Though the Fibonacci series start with 0, 1 (or 1,1) we can take any initial values. Once the first two values are defined the entire Fibonacci series is defined.

One of the interesting properties of Fibonacci series is the ratio between a member and previous member $M_{n+1}/M_n$. where $M_n$ is the nth member and $M_{n+1}$ is the next member. It tends to a value somewhere near 1.618 and this ratio is known as Golden Ratio. Yet more interesting feature is that even if we start the Fibonacci series with a different number other than 0, 1 or 1, 1 the golden ratio remains same. You can experiment this by creating a spread sheet in Excel or Open office.

I have created a document in Google Drive for this purpose. You can click the link below.

https://docs.google.com/spreadsheets/d/1K2O8bYG-a8c_ESQLaJ_NMfc25PbSpUdPiKF_YnU-D3k/edit#gid=1752353278

In the document just enter the first two members of the series. The remaining members are generated automatically. It also show the ratio two adjacent members $R = M_{n+1}/M_n$. Whatever may be the first two members the Ratio R approach golden ratio. To see how this ratio is approaching the golden ratio (G) I have given another column which show the difference between R and G. You can see R is approaching G very quickly. You can also see that R oscillates around the G. That is, if one value of R is above G the next value will be below G.

This means $F_{m+1}/F_m$ tends to a number **x** as m increase. Here $F_m$ is the $m^{th}$ member of Fibonacci series and $F_{m+1}$ is the next member. With this property itself we can find the value of **x**. Suppose two consecutive members of the series be a and b. Then the next member will be a+b.

0, 1, 1, 2, 3 ….. a, b, a+b, …

The golden ratio is b/a and this is same as a+b/b. This means $\frac{b}{a} = \frac{a+b}{b}$

That is: $\frac{b}{a} = \frac{a}{b} + 1$

Since the golden ratio 'x' = b/a we get: $x = \frac{1}{x} + 1$

Solving this quadratic equation we get $x = \frac{1+\sqrt{5}}{2}$ = 1.618033988749895

This means when we multiply a member with a constant we get the next member. So Fibonacci sequence is showing the same properties of a Geometric Sequence. This is an interesting property.

So the next natural question is that does all Geometric Sequences have this property? That is, in a Geometric Sequence

$G_0, G_1, G_2, G_3, \ldots, G_n, G_{n+1}, \ldots$

Can we find some relation between $G_{m+2}$ and sum of $G_{m+1}$ and $G_m$?. The answer is Yes.

Suppose r be the common ratio of a geometric sequence. Then two consecutive members will be:

a, a.r, $ar^2$

We can write $ar^2$ = (a+ar)*k. This means a member in the Geometric Sequence is obtained by adding two previous members and then multiplying with a factor k.

$k = \frac{ar^2}{a+ar}$ *this means* $k = \frac{r^2}{1+r}$ ------------ (1)

In Fibonacci Sequence, $r = \frac{1+\sqrt{5}}{2}$ and using the above equation (1) we get k = 1.

In a Geometric Sequence with r = 3 we get that $k = \frac{3^2}{1+3} = \frac{9}{4} = 2.25$. We can verify this with an arbitrary sequence with r = 3 as below:

1, 3, 9, 27, …

As per the statement above any member is 2.25 times the sum of previous two members. That is, the third member 9 will be 2.25 time (1+3) = 2.25 * 4 = 9 which is true.

If we define the first two terms of the sequence as 2 and 6 and assume k = 2.25 then the subsequent members can be found by adding the previous members and multiplying it with 2.25. If we do this we will get the sequence as:

2, 6, 18, 54, 162, 486, … --------------- (S1)
which is nothing but a Geometric Sequence with r = 3.

But if we define the first two terms of the sequence as 2 and 4 and assume k = 2.25 then the subsequent members can be found by adding the previous members and multiplying it with 2.25. If we do this we will get the sequence as:

2, 4, 13.5, 39.375, 118.96875, 356.2734375, 1069.294922, 3207.528809, 9622.853394, 28868.35995, 86605.23003, …. ------------- (S2)

If we take the ratio of any two adjacent members we get a value near to 3 and as we take higher terms we get more and more close to 3. For example the ratio of $8^{th}$ and $7^{th}$ member = 3207.528809 / 1069.294922 = 2.999667110548571 and the ratio of $9^{th}$ and $8^{th}$ member = 9622.853394 / 3207.528809 = 3.000083231364673 which is closer to 3. As we move to higher order members the ratio comes nearer to 3, but it will not touch 3 in finite steps.

The sequence S1 and S2 are made by assuming the first two members and generating the subsequent members with value of k taken as 2.25. In sequence S1 we got the exact Geometric Sequence with r = 3 but in S2 there is an ‘impurity’. This impurity is nothing but the impurity in selection of first two members.

From equation (1) we know that if **k** = 2.25 then **r** should be 3 and this **r** is kept in the selection of first two members of S1. But this is not kept in S2 and this ‘impurity’ propagates throughout S2: The ratio of two members is coming more and more near to 3 but it never touch 3.